4 Proof of Implicit Function Theorem 2020-21

Notation We are looking at functions $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ with $n > m$. In the proof we will naturally get a decomposition $\mathbb{R}^n = \mathbb{R}^{n-m} \times \mathbb{R}^m$ and we write $\mathbf{x} \in \mathbb{R}^n$ as

$$\mathbf{x} = \begin{pmatrix} \mathbf{v} \\ \mathbf{y} \end{pmatrix} = \left(\mathbf{v}^T, \mathbf{y}^T\right)^T,$$

with $\mathbf{v} \in \mathbb{R}^{n-m}$ and $\mathbf{y} \in \mathbb{R}^m$. This means that $x^i = v^i$ for $1 \le i \le n-m$ while $x^i = y^{i-n+m}$ for $n-m+1 \le i \le n$.

We have previously stated and looked at consequences for surfaces of the result:

Theorem 1 ***Implicit Function Theorem.*** *Suppose that* $\mathbf{f} : U \to \mathbb{R}^m$ *is a* C^1*-function on an open set* $U \subseteq \mathbb{R}^n$, *where* $1 \le m < n$; *there exists* $\mathbf{p} \in U$ *such that* $\mathbf{f}(\mathbf{p}) = \mathbf{0}$ *and the Jacobian matrix* $J\mathbf{f}(\mathbf{p})$ *has full-rank* m.

Suppose that the final m *columns of the Jacobian matrix are linearly independent. Write*

$$\mathbf{p} = \begin{pmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \end{pmatrix}$$

where $\mathbf{p}_0 \in \mathbb{R}^{n-m}$ *and* $\mathbf{p}_1 \in \mathbb{R}^m$.

Then there exists

- *an open set* $V : \mathbf{p}_0 \in V \subseteq \mathbb{R}^{n-m}$,
- *a* C^1*-function* $\phi : V \to \mathbb{R}^m$ *and*
- *an open set* $W : \mathbf{p} \in W \subseteq U \subseteq \mathbb{R}^n$

such for $\mathbf{w} \in W$, *written as*

$$\mathbf{w} = \begin{pmatrix} \mathbf{v} \\ \mathbf{y} \end{pmatrix}$$

with $\mathbf{v} \in \mathbb{R}^{n-m}$ *and* $\mathbf{y} \in \mathbb{R}^m$, *then*

$$\mathbf{f}(\mathbf{w}) = \mathbf{0} \text{ if, and only if, } \mathbf{v} \in V \text{ and } \mathbf{y} = \phi(\mathbf{v}).$$

Briefly, *near a zero other zeros form a surface given by a graph.*

The supposition that the final m columns of the Jacobian matrix $J\mathbf{f}(\mathbf{p})$ form an invertible matrix is satisfied in any given situation by, if necessary, a permutation of the coordinates in $\mathbb{R}^n$. See Section 3 Appendix.

But we can go further in demanding that $J\mathbf{f}(\mathbf{p}) = (A \mid I_m)$ for some $m \times (n-m)$ matrix A. See Section 3 Appendix.

Idea of proof. Induction on $m \geq 1$, the dimension of the image space.

The base case $m = 1$, is Proposition 4 below. In this case we have a scalar-valued function of many variables, $f : \mathbb{R}^n \to \mathbb{R}$. The Jacobian matrix $J\mathbf{f}(\mathbf{p})$ is an $n \times 1$ matrix, and the requirement that it is of full-rank simply means it is non-zero. By permuting the coordinates in $\mathbb{R}^n$ if necessary we assume the **last** entry in $J\mathbf{f}(\mathbf{p})$, i.e. $d_n f(\mathbf{p})$, is non-zero.

We wish to reduce to the case of a scalar-valued function of *one* variable. This is done by looking at f on straight lines $\left(\mathbf{v}^T, y\right)^T$ where $\mathbf{v} \in \mathbb{R}^{n-1}$ is fixed and $y \in \mathbb{R}$ varies. Then define the scalar-valued function of one variable

$$f_{\mathbf{v}}(y) = f\left(\begin{pmatrix} \mathbf{v} \\ y \end{pmatrix}\right),$$

so $f_{\mathbf{v}} : \mathbb{R} \to \mathbb{R}$ and we can use results from second year Real Analysis. In particular the Intermediate Value Theorem: if $g : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and $g(a) < 0 < g(b)$ then there exists $c \in (a, b)$ such that $g(c) = 0$. The complications in the proof arise from showing that $f_{\mathbf{v}}$ takes both positive and negative values. But once we have shown this we conclude that for each $\mathbf{v}$ there exists $y : f_{\mathbf{v}}(y) = 0$. Define the function $\phi_1 : \mathbb{R}^{n-1} \to \mathbb{R}$ by $\phi_1(\mathbf{v}) = y$.

The inductive step Assume the result of the Implicit Function Theorem holds for all appropriate $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^{m-1}$ whenever $n \geq m - 1$.

Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$, with $n \geq m$. We wish to show that the result of the Implicit Function Theorem holds for $\mathbf{f}$.

We can write $\mathbf{f} = (f^1, ..., f^m)^T$. Apply Proposition 4 to the **last** coordinate function $f^m : \mathbb{R}^n \to \mathbb{R}$. This shows that $f^m(\mathbf{w}) = 0$ if and only if $\mathbf{w} = \left(\mathbf{t}^T, \phi_1(\mathbf{t})\right)^T$ for $\mathbf{t} \in \mathbb{R}^{n-1}$ and some function $\phi_1 : \mathbb{R}^{n-1} \to \mathbb{R}$.

Define $\mathbf{g} : \mathbb{R}^{n-1} \to \mathbb{R}^{m-1}$ by its $m-1$ component functions as

$$g^i(\mathbf{t}) = f^i\left(\begin{pmatrix} \mathbf{t} \\ \phi_1(\mathbf{t}) \end{pmatrix}\right),$$

for $1 \leq i \leq m-1$ where $\mathbf{t} \in \mathbb{R}^{n-1}$.

Combining we find that $\mathbf{f}(\mathbf{w}) = \mathbf{0}$ iff

$$\mathbf{w} = \begin{pmatrix} \mathbf{t} \\ \phi_1(\mathbf{t}) \end{pmatrix} \quad \text{for } \mathbf{t} \in \mathbb{R}^{n-1} \text{ and } \quad \mathbf{g}(\mathbf{t}) = \mathbf{0}.$$

In the assumptions of the theorem we are given a point $\mathbf{p} : f(\mathbf{p}) = 0$. So by what we have just shown, there exists $\mathbf{q} \in \mathbb{R}^{n-1} : \mathbf{p} = \left(\mathbf{q}^T, \phi_1(\mathbf{q})\right)^T$ and $\mathbf{g}(\mathbf{q}) = \mathbf{0}$.

Since $\mathbf{g} : \mathbb{R}^{n-1} \to \mathbb{R}^{m-1}$, i.e. the dimension of the image space is $m-1$, we would hope to apply the inductive hypothesis to $\mathbf{g}$ at $\mathbf{q} \in \mathbb{R}^{n-1}$. To use the inductive hypothesis we need that $J\mathbf{g}(\mathbf{q})$ is of full-rank and the tricky part of the proof is showing that this follows from $J\mathbf{f}(\mathbf{p})$ being of full rank.

It transpires that, subject to a permutation of the coordinates in $\mathbb{R}^n$, if you delete the last row and column of the matrix $J\mathbf{f}(\mathbf{p})$ you recover $J\mathbf{g}(\mathbf{q})$. So if $J\mathbf{f}(\mathbf{p})$ is of full-rank it has m linearly independent rows. If you remove one you get $m-1$ linearly independent rows in $J\mathbf{g}(\mathbf{q})$. This is as many rows as $J\mathbf{g}(\mathbf{q})$ has, and so $J\mathbf{g}(\mathbf{q})$ is of full rank.

The induction hypothesis will then give $\mathbf{g}(\mathbf{t}) = \mathbf{0}$ if and only if $\mathbf{t} = \left(\mathbf{v}^T, \mathbf{y}^T\right)^T$ with $\mathbf{v} \in \mathbb{R}^{n-m}$ and $\mathbf{y} = \phi_2(\mathbf{v})$ for some function $\phi_2 : \mathbb{R}^{n-m} \to \mathbb{R}^{m-1}$. That is, $\mathbf{t} = \left(\mathbf{v}^T, \phi_2(\mathbf{v})^T\right)^T$ with $\mathbf{v} \in \mathbb{R}^{n-m}$.

Working back we then find $\mathbf{f}(\mathbf{w}) = \mathbf{0}$ if, and only if,

$$\mathbf{w} = \begin{pmatrix} \mathbf{v} \\ \phi_2(\mathbf{v}) \\ \phi_1\left(\begin{pmatrix} \mathbf{v} \\ \phi_2(\mathbf{v}) \end{pmatrix}\right) \end{pmatrix}$$

with $\mathbf{v} \in \mathbb{R}^{n-m}$.

End of idea

Proof: Base Case

Within the proof of Theorem 1 we will need a Mean Value result for scalar valued functions of several variables. Recall the classical mean value result for $f:[a,b]\to\mathbb{R}$; if continuous on $[a,b]$ and differentiable on (a,b) then there exists $c\in(a,b)$ such that $f(b)-f(a)=f'(c)(b-a)$. What might be hoped for in the case of a scalar-valued function of several variables?

But first, recall the **Chain Rule** in the situation $\mathbb{R}\overset{\mathbf{g}}{\to}\mathbb{R}^n\overset{f}{\to}\mathbb{R}$, when subject to appropriate conditions on the functions,

$$\frac{d}{dt}f(\mathbf{g}(t))=\nabla f(\mathbf{g}(t))\bullet\frac{d}{dt}\mathbf{g}(t)\,. \tag{1}$$

Theorem 2 *Assume $f:U\to\mathbb{R}$ is a scalar-valued C^1-function on an open set $U\subseteq\mathbb{R}^n$. Let $\mathbf{x},\mathbf{y}\in U$ be such that the straight line $\mathbf{x}+t(\mathbf{y}-\mathbf{x}),0\leq t\leq 1$, between $\mathbf{x}$ and $\mathbf{y}$ lies totally within U. Then there exists a point $\mathbf{w}$ on this straight line such that*

$$f(\mathbf{y})-f(\mathbf{x})=\nabla f(\mathbf{w})\bullet(\mathbf{y}-\mathbf{x})\,.$$

Proof The straight line between $\mathbf{x}$ and $\mathbf{y}$ is represented by $\mathbf{x}+t(\mathbf{y}-\mathbf{x})$ for $0\leq t\leq 1$. Define a scalar-valued function of one variable,

$$\psi(t)=f\big(\mathbf{x}+t(\mathbf{y}-\mathbf{x})\big)\quad\text{for}\quad 0\leq t\leq 1.$$

Apply the classical Mean Value Theorem to $\psi(t)$ to find $0<c<1$ such that

$$\psi(1)-\psi(0)=\psi'(c)(1-0)\,,\quad\text{i.e.}\quad f(\mathbf{y})-f(\mathbf{x})=\psi'(c)\,.$$

Apply the Chain rule (1) to differentiate ψ :

$$\frac{d}{dt}\psi(t)=\nabla f\big(\mathbf{x}+t(\mathbf{y}-\mathbf{x})\big)\bullet(\mathbf{y}-\mathbf{x})\,.$$

Thus

$$\psi'(c)=\nabla f(\mathbf{w})\bullet(\mathbf{y}-\mathbf{x})$$

where $\mathbf{w}=\mathbf{x}+c(\mathbf{y}-\mathbf{x})$. These combine to give required result. ■

Lemma 3 *If $g : \mathbb{R}^n \to \mathbb{R}$ is continuous at $\mathbf{a} \in \mathbb{R}^n$ and $g(\mathbf{a}) > 0$ then there exists $\delta > 0$ such that if $\mathbf{x} \in B_\delta(\mathbf{a})$ then $g(\mathbf{x}) > 0$. Similarly, if $g(\mathbf{a}) < 0$ then there exists $\delta > 0$ such that if $\mathbf{x} \in B_\delta(\mathbf{a})$ then $g(\mathbf{x}) < 0$.*

Proof Identical to the $n = 1$ result in MATH20101. If $g(\mathbf{a}) > 0$ choose $\varepsilon = g(\mathbf{a})/2 > 0$ in the definition of g is continuous at $\mathbf{a}$ to find the required δ. If $g(\mathbf{a}) < 0$ choose $\varepsilon = -g(\mathbf{a})/2 > 0$. See Appendix. ■

Proposition 4 *(The $m = 1$ case of the I.F.Th^m) Suppose that $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ is a C^1-function on an open set U and there exists $\mathbf{p} \in U$ such that $f(\mathbf{p}) = \mathbf{0}$ with partial derivative $d_n f(\mathbf{p}) \neq 0$.*
Write $\mathbf{p} = \left(\mathbf{q}^T, c\right)^T$ where $\mathbf{q} \in \mathbb{R}^{n-1}$ and $c \in \mathbb{R}$. Then there exists

- *an open set $A : \mathbf{q} \in A \subseteq \mathbb{R}^{n-1}$,*
- *a C^1-function $\phi : A \to \mathbb{R}$,*
- *an open set $D : \mathbf{p} \in D \subseteq U$,*

such that for $\left(\mathbf{t}^T, y\right)^T \in D$ ($\mathbf{t} \in \mathbb{R}^{n-1}, y \in \mathbb{R}$),

$$f\left(\begin{pmatrix} \mathbf{t} \\ y \end{pmatrix}\right) = 0 \quad \textit{if, and only if,} \quad \mathbf{t} \in A, y = \phi(\mathbf{t}).$$

Further ϕ is a C^1-function on A and satisfies

$$d_j \phi(\mathbf{t}) = -\frac{d_j f\left(\left(\mathbf{t}^T, \phi(\mathbf{t})\right)^T\right)}{d_n f\left(\left(\mathbf{t}^T, \phi(\mathbf{t})\right)^T\right)},$$

for all $\mathbf{t} \in A \subseteq \mathbb{R}^{n-1}$, $1 \leq j \leq n-1$.

Proof Assume $d_n f(\mathbf{p}) > 0$. If $d_n f(\mathbf{p}) < 0$ replace f by $-f$ for $f(\mathbf{x}) = 0$ iff $-f(\mathbf{x}) = 0$.

The proof is split into three parts:

1. existence of ϕ and A**;**
2. ϕ is continuous in A and
3. ϕ is a C^1-function with the partial derivatives shown above**.**

Part 1. Existence of ϕ **and** A**.**

Since f is a C^1-function the derivatives $d_i f(\mathbf{x})$ are continuous. Also $d_n f(\mathbf{p}) > 0$ so we can apply Lemma 3 with $g = d_n f$ to find $\delta > 0$ such that

$$\mathbf{x} \in B_\delta(\mathbf{p}) \implies d_n f(\mathbf{x}) > 0. \tag{2}$$

Define a function of one variable,

$$f_{\mathbf{q}}(y) = f\left(\begin{pmatrix} \mathbf{q} \\ y \end{pmatrix}\right) = f\left(\begin{pmatrix} \mathbf{q} \\ c \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ y - c \end{pmatrix}\right) = f\big(\mathbf{p} + (y-c)\,\mathbf{e}_n\big).$$

Then

$$\begin{aligned} f'_{\mathbf{q}}(y) &= \lim_{h\to 0} \frac{f\big(\mathbf{p} + (y+h-c)\,\mathbf{e}_n\big) - f\big(\mathbf{p} + (y-c)\,\mathbf{e}_n\big)}{h} \\ &= d_n f\big(\mathbf{p} + (y-c)\,\mathbf{e}_n\big). \end{aligned}$$

If we now restrict to $|y - c| < \delta$ then $\mathbf{p} + (y-c)\,\mathbf{e}_n \in B_\delta(\mathbf{p})$ and so $f'_{\mathbf{q}}(y) > 0$ by (2). Since the derivative exists the function $f_{\mathbf{q}}$ *is continuous* and, since the derivative is > 0, we have that $f_{\mathbf{q}}$ is *strictly increasing for* $|y - c| < \delta$.

Note that

$$f_{\mathbf{q}}(c) = f\left(\begin{pmatrix} \mathbf{q} \\ c \end{pmatrix}\right) = f(\mathbf{p}) = 0.$$

So, since $f_{\mathbf{q}}$ is increasing, we can choose c_1 and $c_2 : c - \delta < c_1 < c < c_2 < c + \delta$ (perhaps choosing the mid-way points) for which

$$f_{\mathbf{q}}(c_1) < f_{\mathbf{q}}(c) = 0 < f_{\mathbf{q}}(c_2)\,.$$

In particular $f_{\mathbf{q}}(c_1)$ *is negative* and $f_{\mathbf{q}}(c_2)$ *positive.*

Let

$$\mathbf{a}_1 = \begin{pmatrix} \mathbf{q} \\ c_1 \end{pmatrix},\ \mathbf{a}_2 = \begin{pmatrix} \mathbf{q} \\ c_2 \end{pmatrix} \in \mathbb{R}^n.$$

Note that $\mathbf{a}_1, \mathbf{a}_2 \in B_\delta(\mathbf{p})$.

Then

$$f(\mathbf{a}_1) = f_{\mathbf{q}}(c_1) < 0 \quad \text{and} \quad f(\mathbf{a}_2) = f_{\mathbf{q}}(c_2) > 0,$$

by definition of $f_{\mathbf{q}}$.

Since $f : \mathbb{R}^n \to \mathbb{R}$ is a C^1-function in $B_\delta(\mathbf{p})$ it is continuous within the ball and thus at the two points $\mathbf{a}_1$ and $\mathbf{a}_2$. This means that by Lemma 3 we can find $\delta_1,\ \delta_2 > 0$ such that

$$\mathbf{w} \in B_{\delta_1}(\mathbf{a}_1) \implies f(\mathbf{w}) < 0 \quad \text{while} \quad \mathbf{w} \in B_{\delta_2}(\mathbf{a}_2) \implies f(\mathbf{w}) > 0. \tag{3}$$

By choosing δ_1 and δ_2 sufficiently small we can ensure that $B_{\delta_1}(\mathbf{a}_1), B_{\delta_2}(\mathbf{a}_2) \subseteq B_\delta(\mathbf{p})$.

Let $\delta_0 = \min(\delta_1, \delta_2) > 0$ and set $A = \widehat{B}_{\delta_0}(\mathbf{q})$, an open ball in $\mathbb{R}^{n-1}$ (The $\widehat{\cdots}$ is to show it is a ball in $\mathbb{R}^{n-1}$ not $\mathbb{R}^n$.).

Let $D = A \times (c_1, c_2) \subseteq \mathbb{R}^n$. (Here (c_1, c_2) is an interval, not an ordered pair, and you can think of D as a generalised cylinder)

Note that $\mathbf{q} \in \widehat{B}_{\delta_0}(\mathbf{q})$ and $c \in (c_1, c_2)$ together give

$$\mathbf{p} = \begin{pmatrix} \mathbf{q} \\ c \end{pmatrix} \in \widehat{B}_{\delta_0}(\mathbf{q}) \times (c_1, c_2) = A \times (c_1, c_2) = D.$$

We repeat the above but with $\mathbf{q}$ replaced by any $\mathbf{t} \in \widehat{B}_{\delta_0}(\mathbf{q})$. Define the function of one variable,

$$f_{\mathbf{t}}(y) = f\left(\begin{pmatrix} \mathbf{t} \\ y \end{pmatrix}\right), \quad c_1 \le c \le c_2.$$

Again we can show that with $\mathbf{w}_0 = \left(\mathbf{t}^T,\ c\right)^T$,

$$f'_{\mathbf{t}}(y) = d_n f\left(\mathbf{w}_0 + (y - c)\,\mathbf{e}_n\right) > 0,$$

since $\mathbf{w}_0 + (y - c)\,\mathbf{e}_n \in B_\delta(\mathbf{p})$. Hence $f_{\mathbf{t}}$ is a strictly increasing continuous function.

Set

$$\mathbf{w}_1 = \begin{pmatrix} \mathbf{t} \\ c_1 \end{pmatrix} \quad \text{and} \quad \mathbf{w}_2 = \begin{pmatrix} \mathbf{t} \\ c_2 \end{pmatrix}.$$

Then

$$|\mathbf{w}_1 - \mathbf{a}_1| = \left| \begin{pmatrix} \mathbf{t} \\ c_1 \end{pmatrix} - \begin{pmatrix} \mathbf{q} \\ c_1 \end{pmatrix} \right| = |\mathbf{t} - \mathbf{q}| < \delta_0 \leq \delta_1,$$

since $\mathbf{t} \in A = \widehat{B}_{\delta_0}(\mathbf{q})$. (Make sure you understand why the norm of vectors in $\mathbb{R}^n$ is equal to the norm of vectors in $\mathbb{R}^{n-1}$. Perhaps write them as the root of the sum of squares of the coordinates.) Similarly, $|\mathbf{w}_2 - \mathbf{a}_2| < \delta_2$.

Thus $\mathbf{w}_1 \in B_{\delta_1}(\mathbf{a}_1)$ and $\mathbf{w}_2 \in B_{\delta_2}(\mathbf{a}_2)$. Hence, by (3), $f(\mathbf{w}_1) < 0$ and $f(\mathbf{w}_2) > 0$. Thus $f_{\mathbf{t}}(c_1) = f(\mathbf{w}_1) < 0$ and $f_{\mathbf{t}}(c_2) = f(\mathbf{w}_2) > 0$. Therefore, by the Intermediate Value Theorem applied to $f_{\mathbf{t}}$ on the closed interval $[c_1, c_2]$ there exists $\xi : c_1 < \xi < c_2$ such that $f_{\mathbf{t}}(\xi) = 0$. Since f is *strictly* increasing this value c is *unique*. Define $\phi(\mathbf{t}) = \xi$.

This can be repeated for each $\mathbf{t} \in A$ to define a function $\phi : A \to \mathbb{R}$.

Note that by definition, for $\xi = \phi(\mathbf{t})$,

$$0 = f_{\mathbf{t}}(\xi) = f\left(\begin{pmatrix} \mathbf{t} \\ \xi \end{pmatrix}\right) = f\left(\begin{pmatrix} \mathbf{t} \\ \phi(\mathbf{t}) \end{pmatrix}\right).$$

Part 2. ϕ is continuous in A.

Let $\mathbf{t} \in A \subseteq \mathbb{R}^{n-1}$ be given. We will show that $\lim_{\mathbf{s}\to\mathbf{0}} \phi(\mathbf{t} + \mathbf{s}) = \phi(\mathbf{t})$. Assume $\mathbf{s} \in \mathbb{R}^{n-1}$ is such that $\mathbf{t} + \mathbf{s} \in A$, possible since A is an open set.

Recall that $\phi(\mathbf{t} + \mathbf{s})$ is defined to satisfy

$$0 = f\left(\begin{pmatrix} \mathbf{t} + \mathbf{s} \\ \phi(\mathbf{t} + \mathbf{s}) \end{pmatrix}\right).$$

Let $\mathbf{s} \to \mathbf{0}$ and use the continuity of f to say

$$\begin{aligned} 0 &= \lim_{\mathbf{s}\to\mathbf{0}} f\left(\begin{pmatrix} \mathbf{t} + \mathbf{s} \\ \phi(\mathbf{t} + \mathbf{s}) \end{pmatrix}\right) = f\left(\begin{pmatrix} \lim_{\mathbf{s}\to\mathbf{0}} (\mathbf{t} + \mathbf{s}) \\ \lim_{\mathbf{s}\to\mathbf{0}} \phi(\mathbf{t} + \mathbf{s}) \end{pmatrix}\right) \\ &= f\left(\begin{pmatrix} \mathbf{t} \\ \lim_{\mathbf{s}\to\mathbf{0}} \phi(\mathbf{t} + \mathbf{s}) \end{pmatrix}\right). \end{aligned}$$

Yet $\phi(\mathbf{t})$ is defined to satisfy

$$0 = f\left(\left(\begin{array}{c}\mathbf{t}\\ \phi(\mathbf{t})\end{array}\right)\right),$$

and, for a given $\mathbf{t}$, $\phi(\mathbf{t})$ is *unique*, so we must have $\lim_{\mathbf{s}\to\mathbf{0}} \phi(\mathbf{t}+\mathbf{s}) = \phi(\mathbf{t})$, i.e. ϕ is continuous at $\mathbf{t}$ and thus on A.

(Never lose sight of the fact that the vectors here are in $\mathbb{R}^{n-1}$).

Part 3. ϕ is a C^1-function.

We will show that the partial derivatives $d_j\phi$ *exist* throughout $A = \widehat{B}_{\delta_0}(\mathbf{q})$ for each $1 \le j \le n-1$ and are *continuous*. This is the definition of a C^1-function.

Let $\mathbf{t} \in \widehat{B}_{\delta_0}(\mathbf{q})$ be given. To calculate the j-th partial derivative $d_j\phi$ we need to look at the ratio

$$\frac{\phi(\mathbf{t}+h\widehat{\mathbf{e}}_j) - \phi(\mathbf{t})}{h}$$

as $h \to 0$. Here $\widehat{\mathbf{e}}_j$ is a standard basis vector of $\mathbb{R}^{n-1}$ written with a $\widehat{\ }$ to differentiate it from basis vectors $\mathbf{e}_j$ of $\mathbb{R}^n$. Since the ball $\widehat{B}_{\delta_0}(\mathbf{q})$ is an open set, we have $\mathbf{t}+h\widehat{\mathbf{e}}_j \in \widehat{B}_{\delta_0}(\mathbf{q})$ for $|h| < \eta$ with some $\eta > 0$. By the definition of ϕ we have

$$f\left(\left(\begin{array}{c}\mathbf{t}\\ \phi(\mathbf{t})\end{array}\right)\right) = 0 \quad\text{and}\quad f\left(\left(\begin{array}{c}\mathbf{t}+h\widehat{\mathbf{e}}_j\\ \phi(\mathbf{t}+h\widehat{\mathbf{e}}_j)\end{array}\right)\right) = 0.$$

Rewrite these with $y = \phi(\mathbf{t})$ so

$$f\left(\left(\begin{array}{c}\mathbf{t}\\ y\end{array}\right)\right) = 0 \quad\text{and}\quad f\left(\left(\begin{array}{c}\mathbf{t}+h\widehat{\mathbf{e}}_j\\ y+u\end{array}\right)\right) = 0,$$

where $u = \phi(\mathbf{t}+h\widehat{\mathbf{e}}_j) - \phi(\mathbf{t})$. Subtracting equal values gives

$$f\left(\left(\begin{array}{c}\mathbf{t}+h\widehat{\mathbf{e}}_j\\ y+u\end{array}\right)\right) - f\left(\left(\begin{array}{c}\mathbf{t}\\ y\end{array}\right)\right) = 0.$$

Now apply the Mean Value Theorem, 2, but first note that the straight line between $\left(\mathbf{t}^T, y\right)^T$ and $\left((\mathbf{t}+h\widehat{\mathbf{e}}_j)^T,\, y+u\right)^T$ is given by

$$\left\{\begin{pmatrix} \mathbf{t}+sh\widehat{\mathbf{e}}_j \\ y+su \end{pmatrix} : 0 \le s \le 1\right\}.$$

The difference between the ends of the line can be easily expressed in terms of basis vectors (of $\mathbb{R}^n$) as

$$\begin{pmatrix} \mathbf{t}+h\widehat{\mathbf{e}}_j \\ y+u \end{pmatrix} - \begin{pmatrix} \mathbf{t} \\ y \end{pmatrix} = \begin{pmatrix} h\widehat{\mathbf{e}}_j \\ u \end{pmatrix} = h\mathbf{e}_j + u\mathbf{e}_n.$$

Aside Make sure this part is understood, how we go from basis vectors in $\mathbb{R}^{n-1}$ to basis vectors in $\mathbb{R}^n$:

$$\begin{pmatrix} h\widehat{\mathbf{e}}_j \\ u \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ h \\ \vdots \\ 0 \end{pmatrix} \\ u \end{pmatrix} = h\begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 1 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} + u\begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ \vdots \\ \vdots \\ 1 \end{pmatrix} = h\mathbf{e}_j + u\mathbf{e}_n.$$

End of Aside

The Mean Value Theorem asserts that there exists $0 < \sigma < 1$ such that

$$0 = f\left(\begin{pmatrix} \mathbf{t}+h\widehat{\mathbf{e}}_j \\ y+u \end{pmatrix}\right) - f\left(\begin{pmatrix} \mathbf{t} \\ y \end{pmatrix}\right) = \nabla f(\mathbf{w}) \bullet (h\mathbf{e}_j + u\mathbf{e}_n), \qquad (4)$$

where $\mathbf{w} = \left((\mathbf{t}+\sigma h\widehat{\mathbf{e}}_j)^T, y+\sigma u\right)^T$.

Next, $\nabla f(\mathbf{w}) \bullet \mathbf{e}_j$ is the j-th coordinate of $\nabla f(\mathbf{w})$, which is $\partial f(\mathbf{w})/\partial x^j = d_j f(\mathbf{w})$. Similarly, $\nabla f(\mathbf{w}) \bullet \mathbf{e}_n = d_n f(\mathbf{w})$. Hence (4) becomes

$$0 = hd_j f(\mathbf{w}) + ud_n f(\mathbf{w}).$$

This rearranges as

$$\frac{u}{h} = -\frac{d_j f(\mathbf{w})}{d_n f(\mathbf{w})},$$

i.e.

$$\frac{\phi(\mathbf{t} + h\widehat{\mathbf{e}}_j) - \phi(\mathbf{t})}{t} = -\frac{d_j f\left(\left((\mathbf{t} + \sigma h\widehat{\mathbf{e}}_j)^T, y + \sigma u\right)^T\right)}{d_n f\left(\left((\mathbf{t} + \sigma h\widehat{\mathbf{e}}_j)^T, y + \sigma u\right)^T\right)}.$$

Now let $h \to 0$. By the continuity of ϕ (part 2) we have

$$u = \phi(\mathbf{t} + h\widehat{\mathbf{e}}_j) - \phi(\mathbf{t}) \longrightarrow \phi(\mathbf{t}) - \phi(\mathbf{t}) = 0.$$

Thus

$$\begin{pmatrix} \mathbf{t} + \sigma h\widehat{\mathbf{e}}_j \\ y + \sigma u \end{pmatrix} \longrightarrow \begin{pmatrix} \mathbf{t} \\ y \end{pmatrix} = \begin{pmatrix} \mathbf{t} \\ \phi(\mathbf{t}) \end{pmatrix},$$

by definition of y, as $h \to 0$.

Now use the fact that f is a C^1-function which means that each $d_k f(\mathbf{x})$ is continuous, $1 \le k \le n$. With $k = j$ and n we deduce, (with the Quotient Rule for limits and the assumption that $d_n f(\mathbf{a}) \ne 0$)

$$\begin{aligned}
\lim_{h\to 0} \frac{\phi(\mathbf{t} + h\widehat{\mathbf{e}}_j) - \phi(\mathbf{t})}{h} &= -\frac{\lim_{h\to 0} d_j f\left(\left((\mathbf{t} + \sigma h\widehat{\mathbf{e}}_j)^T, y + \sigma u\right)^T\right)}{\lim_{h\to 0} d_n f\left(\left((\mathbf{t} + \sigma h\widehat{\mathbf{e}}_j)^T, y + \sigma u\right)^T\right)} \\
&= -\frac{d_j f\left(\lim_{t\to 0}\left((\mathbf{t} + \sigma h\widehat{\mathbf{e}}_j)^T, y + \sigma u\right)^T\right)}{d_n f\left(\lim_{t\to 0}\left((\mathbf{t} + \sigma h\widehat{\mathbf{e}}_j)^T, y + \sigma u\right)^T\right)} \\
&= -\frac{d_j f\left((\mathbf{t}^T, y)^T\right)}{d_n f\left((\mathbf{t}^T, y)^T\right)}. \qquad (5)
\end{aligned}$$

That the limit exists means that $d_j\phi(\mathbf{t})$ exists, and since $\mathbf{t}$ was arbitrary it exists on $\widehat{B}_{\delta_0}(\mathbf{q})$.

Further, since f is a C^1-function the right hand side of (5) is continuous, and thus $d_j\phi$ are continuous functions of $\mathbf{t}$ for all $1 \le j \le n-1$. ■

Proof: Inductive Step

Proposition 5 ***The inductive step*** *Suppose that the Implicit Function Theorem holds for any appropriate function* $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^{m-1}$*, at* $\mathbf{p} \in U$ *with* $J\mathbf{f}(\mathbf{p})$ *of the form* $(A \,|\, I_{m-1})$*, for any* $n > m - 1$*. Then the Theorem holds for any appropriate function* $\mathbf{f} : U \to \mathbb{R}^m$*, at* $\mathbf{p} \in U \subseteq \mathbb{R}^n$ *with* $J\mathbf{f}(\mathbf{p})$ *of the form* $(A \,|\, I_m)$*, for any* $n > m$*.*

Proof Suppose that $\mathbf{f} : U \to \mathbb{R}^m$ is a C^1-function on an open set $U \subseteq \mathbb{R}^n$, where $1 \leq m < n$ and there exists $\mathbf{p} \in U$ such that $\mathbf{f}(\mathbf{p}) = \mathbf{0}$ and $J\mathbf{f}(\mathbf{p}) = (A \,|\, I_m)$ for some $m \times (n-m)$ matrix A.

At this point it becomes notationally easier to consider elements of $\mathbf{w} \in \mathbb{R}^n$ as elements of $\mathbb{R}^{n-m} \times \mathbb{R}^m$, written as $\mathbf{w} = \left(\mathbf{v}^T, \mathbf{y}^T\right)^T$, where $\mathbf{v} \in \mathbb{R}^{n-m}$, $\mathbf{y} \in \mathbb{R}^m$. So

$$\mathbf{w} = \left(\mathbf{v}^T, \mathbf{y}^T\right)^T = \left(v^1, v^2, ..., v^{n-m}, y^1, y^2, ..., y^m\right)^T. \tag{6}$$

The identity matrix in $J\mathbf{f}(\mathbf{p}) = (A \,|\, I_m)$ represents

$$I_m = \left(\frac{\partial f^i(\mathbf{p})}{\partial y^j}\right)_{\substack{1 \leq i \leq m, \\ 1 \leq j \leq m}},$$

that is

$$\frac{\partial f^i(\mathbf{p})}{\partial y^j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \tag{7}$$

Apply Proposition 4 to the last scalar-valued component function $f^m : U \to \mathbb{R}$, $U \subseteq \mathbb{R}^n$. Write $\mathbf{p} = \left(\mathbf{q}^T, c\right)^T$ where $\mathbf{q} \in \mathbb{R}^{n-1}$ and $c \in \mathbb{R}$. Then there exists

- an open set $V_1 : \mathbf{q} \in V_1 \subseteq \mathbb{R}^{n-1}$,
- a C^1-function $\phi_1 : V_1 \to \mathbb{R}$,
- an open set $W_1 : \mathbf{p} \in W_1 \subseteq U$,

such that for $\left(\mathbf{t}^T, y\right)^T \in W_1$ ($\mathbf{t} \in \mathbb{R}^{n-1}, y \in \mathbb{R}$),

$$f^m\left(\begin{pmatrix} \mathbf{t} \\ y \end{pmatrix}\right) = 0 \quad \text{if, and only if,} \quad \mathbf{t} \in V_1,\ y = \phi_1(\mathbf{t}).$$

In particular, $f^m(\mathbf{p}) = 0$ implies $\mathbf{q} \in V_1, c = \phi_1(\mathbf{q})$.

From the remaining components of $\mathbf{f}$ define new functions on V_1 by

$$g^i(\mathbf{t}) = f^i\left(\begin{pmatrix} \mathbf{t} \\ \phi_1(\mathbf{t}) \end{pmatrix}\right),$$

for $\mathbf{t} \in V_1$, $1 \le i \le m-1$. Define $\mathbf{g} : V_1 \to \mathbb{R}^{m-1}$ by $\mathbf{g} = \left(g^1, g^2, ..., g^{m-1}\right)^T$. Then

$$\mathbf{f}(\mathbf{w}) = \mathbf{0} \iff \mathbf{w} = \begin{pmatrix} \mathbf{t} \\ \phi_1(\mathbf{t}) \end{pmatrix} \quad \text{and} \quad \mathbf{g}(\mathbf{t}) = \mathbf{0}.$$

Note that for each $1 \le i \le m-1$ we have

$$g^i(\mathbf{q}) = f^i\left(\begin{pmatrix} \mathbf{q} \\ \phi_1(\mathbf{q}) \end{pmatrix}\right) = f^i\left(\begin{pmatrix} \mathbf{q} \\ c \end{pmatrix}\right) = f^i(\mathbf{p}) = 0.$$

since $\phi_1(\mathbf{q}) = c$. Hence $\mathbf{g}(\mathbf{q}) = 0$.

The g^i are C^1-functions. To see this we note that for $1 \le i \le m-1$ we have a composition of functions

$$\mathbf{t} \mapsto \begin{pmatrix} \mathbf{t} \\ \phi_1(\mathbf{t}) \end{pmatrix} \mapsto f^i\left(\begin{pmatrix} \mathbf{t} \\ \phi_1(\mathbf{t}) \end{pmatrix}\right).$$

Temporarily define

$$\mathbf{h} : V_1 \subseteq \mathbb{R}^{n-1} \to \mathbb{R}^n : \mathbf{t} \mapsto \begin{pmatrix} \mathbf{t} \\ \phi_1(\mathbf{t}) \end{pmatrix}, \tag{8}$$

in which case $g^i = f^i \circ \mathbf{h}$. Here $\mathbf{h}$ is a function of $\mathbf{t} \in \mathbb{R}^{n-1}$, but think of $\mathbf{f}$ as a function of $\mathbf{w} \in \mathbb{R}^n$. The Chain Rule then gives, for $1 \le i \le m-1, 1 \le j \le n-1$,

$$\frac{\partial g^i}{\partial t^j}(\mathbf{t}) = \frac{\partial f^i \circ \mathbf{h}}{\partial t^j}(\mathbf{t}) = \sum_{k=1}^{n} \frac{\partial f^i}{\partial w^k}(\mathbf{h}(\mathbf{t})) \frac{\partial h^k}{\partial t^j}(\mathbf{t}), \tag{9}$$

for $\mathbf{t} \in V_1$. From it's definition, (8), $h^k(\mathbf{t}) = t^k$ if $1 \leq k \leq n-1$ while $h^n(\mathbf{t}) = \phi_1(\mathbf{t})$. Thus, for $1 \leq k \leq n-1$,

$$\frac{\partial h^k}{\partial t^j}(\mathbf{v}) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j, \end{cases}$$

while, for $k = n$,

$$\frac{\partial h^n}{\partial t^j}(\mathbf{t}) = \frac{\partial \phi_1}{\partial t^j}(\mathbf{t}),$$

for any $\mathbf{t}$.

Hence (9) reduces to

$$\frac{\partial g^i}{\partial t^j}(\mathbf{t}) = \frac{\partial f^i \circ \mathbf{h}}{\partial t^j}(\mathbf{t}) = \frac{\partial f^i}{\partial w^j}(\mathbf{h}(\mathbf{t})) + \frac{\partial f^i}{\partial w^n}(\mathbf{h}(\mathbf{t}))\frac{\partial \phi_1}{\partial t^j}(\mathbf{t}), \tag{10}$$

for $1 \leq i \leq m-1, 1 \leq j \leq n-1$.

Since f and ϕ_1 are C^1-functions the derivatives on the right hand side of (10) are continuous, and this shows that the derivatives of $\mathbf{g}$ are continuous, therefore $\mathbf{g}$ is a C^1-function.

$Jg(\mathbf{q})$ is of full-rank To see this, choose $\mathbf{v} = \mathbf{q}$ in (10), noting that $\mathbf{h}(\mathbf{q}) = \mathbf{p}$. Recall from (6) that we found it notationally convenient to write $\mathbf{w}$ in coordinates as $(t^1, t^2, ..., t^{n-m}, y^1, y^2, ..., y^m)$, so $w^n = y^m$. Then from (7)

$$\frac{\partial f^i}{\partial w^n}(\mathbf{p}) = \frac{\partial f^i}{\partial y^m}(\mathbf{p}) = 0$$

since $i \neq m$. Thus (10) reduces to

$$\frac{\partial g^i}{\partial t^j}(\mathbf{q}) = \frac{\partial f^i}{\partial w^j}(\mathbf{p}),$$

for $1 \leq i \leq m-1, 1 \leq j \leq n-1$.

These are elements of Jacobian matrices and equality shows that the Jacobian matrix $J\mathbf{g}(\mathbf{q})$ can be obtained from the matrix $J\mathbf{f}(\mathbf{p})$ by deleting the last row and column. Hence $J\mathbf{g}(\mathbf{q}) = (A' \mid I_{m-1})$ for some $(m-1)\times(n-m)$ matrix A' and in particular it is of full-rank.

Induction Thus $J\mathbf{g}(\mathbf{q})$ is of the required form to apply the inductive hypothesis. Write $\mathbf{q} \in \mathbb{R}^{n-1}$ as $\mathbf{q} = \left(\mathbf{q}_1^T, \mathbf{q}_2^T\right)^T$ where $\mathbf{q}_1 \in \mathbb{R}^{n-m}$ and $\mathbf{q}_2 \in \mathbb{R}^{m-1}$. By the inductive hypothesis applied to $\mathbf{g}$ at $\mathbf{q}$, there exists

- an open set $V_2 : \mathbf{q}_1 \in V_2 \subseteq \mathbb{R}^{n-m}$,
- a C^1-function $\phi_2 : V_2 \to \mathbb{R}^{m-1}$ and
- an open set $W_2 : \mathbf{q} \in W_2 \subseteq V_1 \subseteq \mathbb{R}^{n-1}$

such for $\mathbf{t} \in W_2$, written as

$$\mathbf{t} = \begin{pmatrix} \mathbf{v} \\ \mathbf{k} \end{pmatrix},$$

where $\mathbf{v} \in \mathbb{R}^{n-m}$ and $\mathbf{k} \in \mathbb{R}^{m-1}$,

$$\mathbf{g}(\mathbf{t}) = \mathbf{0} \quad \text{if, and only if,} \quad \mathbf{v} \in V_2 \ \text{ and } \mathbf{k} = \phi_2(\mathbf{v}).$$

Combining,

$$\begin{aligned} \mathbf{f}(\mathbf{w}) = \mathbf{0} \iff \mathbf{w} &= \begin{pmatrix} \mathbf{t} \\ \phi_1(\mathbf{t}) \end{pmatrix}, \ \mathbf{t} \in V_1 \quad \text{and} \quad \mathbf{g}(\mathbf{t}) = \mathbf{0}, \\ \iff \mathbf{w} &= \begin{pmatrix} \mathbf{t} \\ \phi_1(\mathbf{t}) \end{pmatrix} \quad \text{and} \quad \mathbf{t} = \begin{pmatrix} \mathbf{v} \\ \phi_2(\mathbf{v}) \end{pmatrix}, \ \mathbf{v} \in V_2. \end{aligned}$$

That is, $f(\mathbf{w}) = 0$ iff

$$\mathbf{w} = \begin{pmatrix} \mathbf{v} \\ \phi_2(\mathbf{v}) \\ \phi_1\left(\begin{pmatrix} \mathbf{v} \\ \phi_2(\mathbf{v}) \end{pmatrix}\right) \end{pmatrix},$$

with $\mathbf{v} \in \mathbf{V}_2$. This can be written as required for the statement of the Theorem as

$$\mathbf{w} = \begin{pmatrix} \mathbf{v} \\ \phi(\mathbf{v}) \end{pmatrix} \quad \text{with} \quad \phi(\mathbf{x}) = \begin{pmatrix} \phi_2(\mathbf{v}) \\ \phi_1\left(\begin{pmatrix} \mathbf{v} \\ \phi_2(\mathbf{v}) \end{pmatrix}\right) \end{pmatrix}.$$

■

Appendix for Section 4

Lemma 3 *If* $g : \mathbb{R}^n \to \mathbb{R}$ *is continuous at* $\mathbf{a} \in \mathbb{R}^n$ *and* $g(\mathbf{a}) > 0$ *then there exists* $\delta > 0$ *such that if* $\mathbf{x} \in B_\delta(\mathbf{a})$ *then* $g(\mathbf{x}) > 0$*. Similarly, if* $g(\mathbf{a}) < 0$ *then there exists* $\delta > 0$ *such that if* $\mathbf{x} \in B_\delta(\mathbf{a})$ *then* $g(\mathbf{x}) < 0$*.*

Proof The assumption that $g : \mathbb{R}^n \to \mathbb{R}$ is continuous at $\mathbf{a} \in \mathbb{R}^n$ means

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall \mathbf{x} \in \mathbb{R}^n,\ \mathbf{x} \in B_\delta(\mathbf{a}) \implies |g(\mathbf{x}) - g(\mathbf{a})| < \varepsilon.$$

If $g(\mathbf{a}) > 0$ choose $\varepsilon = g(\mathbf{a})/2$ in the definition to find $\delta > 0$ such that

$$\begin{aligned}
\mathbf{x} \in B_\delta(\mathbf{a}) \quad &\implies \quad |g(\mathbf{x}) - g(\mathbf{a})| < \frac{g(\mathbf{a})}{2} \\
&\implies \quad -\frac{g(\mathbf{a})}{2} < g(\mathbf{x}) - g(\mathbf{a}) < \frac{g(\mathbf{a})}{2} \\
&\implies \quad -\frac{g(\mathbf{a})}{2} < g(\mathbf{x}) - g(\mathbf{a}) \\
&\implies \quad g(\mathbf{x}) > \frac{g(\mathbf{a})}{2} > 0.
\end{aligned}$$

If $g(\mathbf{a}) < 0$ choose $\varepsilon = -g(\mathbf{a})/2 > 0$ in the definition to find $\delta > 0$ such that

$$\begin{aligned}
\mathbf{x} \in B_\delta(\mathbf{a}) \quad &\implies \quad |g(\mathbf{x}) - g(\mathbf{a})| < -\frac{g(\mathbf{a})}{2} \\
&\implies \quad \frac{g(\mathbf{a})}{2} < g(\mathbf{x}) - g(\mathbf{a}) < -\frac{g(\mathbf{a})}{2} \\
&\implies \quad g(\mathbf{x}) - g(\mathbf{a}) < -\frac{g(\mathbf{a})}{2} \\
&\implies \quad g(\mathbf{x}) < \frac{g(\mathbf{a})}{2} < 0.
\end{aligned}$$

■

In the proof of Proposition 5 we might have given more details at one point. The Jacobian matrix of $\mathbf{f}(\mathbf{w})$ at $\mathbf{w} = \mathbf{p}$ is a matrix of partial derivatives

$$\left(\frac{\partial f^i(\mathbf{p})}{\partial w^j} \right)_{\substack{1 \le i \le m, \\ 1 \le j \le n}}. \tag{11}$$

Yet at this point it becomes notationally easier to consider elements $\mathbf{w} \in \mathbb{R}^n$ as elements of $\mathbb{R}^{n-m} \times \mathbb{R}^m$, written as $\mathbf{w} = \left(\mathbf{v}^T, \mathbf{y}^T\right)^T$, where $\mathbf{v} \in \mathbb{R}^{n-m}, \mathbf{y} \in \mathbb{R}^m$. So

$$\mathbf{w} = \begin{pmatrix} \mathbf{v} \\ \mathbf{y} \end{pmatrix} = \left(v^1, v^2, ..., v^{n-m}, y^1, y^2, ..., y^m\right)^T$$

In this notation, the matrix in (11) is written as

$$\left(\left(\frac{\partial f^i(\mathbf{p})}{\partial v^j} \right)_{\substack{1 \le i \le m, \\ 1 \le j \le n-m}} \middle| \left(\frac{\partial f^i(\mathbf{p})}{\partial y^j} \right)_{\substack{1 \le i \le m, \\ 1 \le j \le m}} \right).$$

Yet we are assuming $J\mathbf{f}(\mathbf{p})$ is of the form $(A \,|\, I_m)$, so

$$\left(\frac{\partial f^i(\mathbf{p})}{\partial y^j} \right)_{\substack{1 \le i \le m, \\ 1 \le j \le m}} = I_m,$$

that is

$$\frac{\partial f^i(\mathbf{p})}{\partial y^j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$